Three-Dimensional Supersonic Nozzle Flowfield Calculations

VICTOR H. RANSOM,* H. DOYLE THOMPSON,† AND JOE D. HOFFMAN‡
Purdue University, Lafayette, Ind.

A second-order numerical method of characteristics has been developed for the solution of three-dimensional supersonic flows. The method has been programed for the CDC 6500 and IBM 7094 computers for internal flows. Solutions are presented for several heretofore unsolved three-dimensional nozzle flow problems. These include: transition from an axisymmetric inlet to an elliptical cross section, transition from an axisymmetric inlet to a near rectangular cross section, and nonsymmetric inlet flow into an axisymmetric nozzle. The results of these calculations reveal the complexity of the pressure field in flows of this type and show that small deviations from axial symmetry produce gross three-dimensional effects that can only be determined using a truly three-dimensional calculation scheme.

Nomenclature

L	=	length along flow direction
P,PC	=	pressure and stagnation pressure, respectively
RC	=	throat circular radius of curvature
RT	=	throat radius
X	=	cartesian coordinate in flow direction
Y,Z	=	cartesian coordinates normal to flow direction
α	=	thrust misalignment angle
ϕ	=	polar angle

Introduction

HERETOFORE, most thrust nozzles have been axisymmetric. This has been due not only to the relative ease of fabrication of axisymmetric shapes, but also to the inability to analyze and, therefore, design three-dimensional nozzles. However, the demand for increased performance in future propulsion applications may dictate consideration of truly three-dimensional thrust nozzles. Typical examples are: the integration of the exhaust expansion system with the vehicle airframe for drag reduction, rocket nozzles for use in systems having nonaxisymmetric nozzle exit constraints, etc.

In many cases three-dimensional flow effects are present even though the boundaries are symmetric. These cases occur when nonsymmetric inlet conditions exist or when nonsymmetric disturbances are generated within the nozzle. The first condition results in thrust misalignment and may arise from a number of sources including the use of multiple nozzles for solid rocket motors or nozzles gimbaled about a throat joint. The second condition occurs when secondary streams are injected into the nozzle or the nozzle boundary is deformed for the purpose of achieving thrust vector control.

For the first time, it is now possible to investigate analytically the characteristics of many nonsymmetric nozzle flows. A new three-dimensional method of characteristics has been developed which has second-order accuracy.[1,2] This method has been programed for both the CDC 6500 and IBM 7094 computers, and results which have been obtained with these programs are presented herein. The primary objective of this paper is to illustrate the potential of the computational method and the computer program by presenting solutions for several three-dimensional nozzle flows. The method can be extended, by modification of boundary and initial conditions, to calculate external supersonic flows, exhaust plumes, and a variety of other important three-dimensional flows.

Many numerical schemes have been proposed for the calculation of three-dimensional supersonic flows (reviews are given in Refs. 3–5), but few have actually been applied. Only one other attempt to produce a numerical algorithm for internal flows has been reported in the U.S. literature. This effort was conducted by Reed[6-8]; however, the effort was discontinued prior to obtaining an operational program. Reed employed the hexahedral method of bicharacteristics, which was originally proposed by Thornhill.[9] Several operational computer programs for three-dimensional external flows and plumes have been produced in this country.[10-14] Both ordinary finite difference schemes and method of characteristics schemes have been employed in these programs. The schemes have differed widely in their applicability to particular problems and several have been shown to yield results in good agreement with experimental data. This is particularly true for the flow around blunted cones at angles of attack. However, all the schemes share the characteristic that only first-order accuracy is achieved and, consequently, convergence to the correct solution with reduction in step size is linear. For high-gradient flows such as exist near the throat of a rocket nozzle, first-order schemes require very small step sizes for good results. In three-dimensional flows the number of computed points is proportional to the inverse of the step size cubed; consequently, the limited capacity of the computer places a lower limit on the step size.

The present approach maintains second-order numerical accuracy and, thus, convergence with reduction in step size is quadratic. The method is based on the theoretical work of Butler,[15] and the development of the numerical scheme is reported in Refs. 1, 2, and 16.

Gas Dynamic Model and Integration Scheme

The continuity equation, the Euler momentum equations, the equation for entropy conservation along streamlines, and the thermal and caloric equations of state are applied to the inviscid, steady, adiabatic flow with frozen or equilibrium chemical composition, and smooth initial data and boundaries. The existence of shock waves is implicitly excluded (but could be included as a mathematical boundary across which the Rankine-Hugoniot conditions are applied). Nonhomentropic and nonisoenergetic flows are permitted; however, the gradients in entropy and stagnation enthalpy are everywhere nor-

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* Research Associate in Mechanical Engineering; now with the Aerojet-General Corporation, Sacramento, Calif. Member AIAA.

† Associate Professor of Mechanical Engineering; currently on leave at Pratt and Whitney Aircraft, E. Hartford, Conn. Member AIAA.

‡ Associate Professor of Mechanical Engineering. Member AIAA.

mal to the streamlines. The equations of motion are a system of quasilinear, hyperbolic, partial differential equations with the three space coordinates as independent variables. The hyperbolic property permits the use of a marching type integration scheme in the flow-wise direction, since each point in the flow has a finite zone of dependence. A problem is properly posed when initial data are given over a space-like initial surface and the initial surface is adjoined by time-like surfaces on which boundary conditions are known.

In the application of the method of characteristics to two-dimensional hyperbolic problems, second-order accuracy can be easily achieved by the use of any explicit second-order integration scheme, such as the modified Euler scheme. This is possible because the characteristic compatibility equations involve directional differentiation in one lower dimension (total derivatives in this case) and, as a result, the derivatives at the unknown point can be expressed in terms of the dependent variables of the problem. In three-dimensional problems the situation is more complicated. The characteristic compatibility equations contain at least two independent directional derivatives; consequently, when the compatibility equations are written in terms of differentials along the bicharacteristic direction, some of the terms of the equations contain derivatives with respect to a second independent direction. These terms are referred to as cross-derivative terms. Ordinary explicit integration schemes cannot achieve second-order accuracy because the cross-derivatives cannot be evaluated at the unknown point. Butler[15] developed an explicit integration scheme for certain types of hyperbolic differential equations which obtains second-order accuracy. Elliot[17] and Richardson[18] applied this technique successfully to the problem of two-dimensional, unsteady flow. In the present research a similar scheme has been developed for the equations of three-dimensional steady flow.

In general, the scheme consists of selecting the number and orientation of the bicharacteristic compatibility equations in such a way that the terms containing cross-derivatives at the unknown point appear in two distinct groupings which are common to all the equations. The two common terms evaluated at the unknown point are treated as additional unknowns, and sufficient independent differential relations are found to enable evaluation, or equivalently, algebraic elimination, of these terms. The two additional differential relations are an additional bicharacteristic compatibility relation and an independent noncharacteristic linear combination of the governing differential equations which is applied along the streamline direction.

Computer Program§

The numerical algorithm has been programmed for both the CDC 6500 and the IBM 7094 computers using Fortran IV language. Various options permit the solution of a wide range of exhaust nozzle problems. The IBM 7094 version of the program is available from the Aero Propulsion Laboratory, Wright-Patterson Air Force Base, Ohio.

The gas properties subroutine which is provided with the program has two options; either a calorically and thermally perfect gas for nonisoenergetic and nonhomentropic flow, or a real gas having frozen or equilibrium chemical composition for an isoenergetic and homentropic flow. In either case, pressure, stagnation pressure, and stagnation enthalpy are used as independent variables for establishing density, sound speed, and velocity magnitude at any point in the flow.

The flow is permitted to have up to eight planes of symmetry, and the program calculates only one sector of the flow. The remaining sectors are found by reflection about the plane of symmetry. The nozzle geometries provided include axisymmetric conical and parabolic nozzles having circular arc entrances, and nozzles having "super-elliptical" cross sections defined by

$$(Y/A)^{NY} + (Z/B)^{NZ} = 1 \tag{1}$$

where the exponents NY and NZ, and the intercepts A and B, are differentiable functions of the X-coordinate direction, which is the direction of integration. "Super-elliptical" cross sections result when the exponents have values greater than 2.0. Using these cross sections it is possible to generate a wide variety of nozzle shapes having smooth transition from circular to near rectangular cross section. The program also has available a link for a user-supplied subprogram which can be used to define the nozzle boundary. The only restrictions on the boundary are that it be a continuous and continuously differentiable surface so that the outer normal is everywhere unique.

Much of the character of a flow is determined by the conditions specified on the initial-value surface. For example, whether or not the flow is homentropic and/or isoenergetic depends upon whether these conditions exist on the initial-value surface. The initial-value data specify P, PT, the velocity components, and stagnation enthalpy over a planar surface normal to the X-coordinate direction. The initial data must adjoin the boundaries and satisfy the boundary conditions at the common points, and they must be symmetric about the planes of symmetry.

The initial-value surface options which are included with the program are: uniform flow, homentropic point source flow, and an axisymmetric nonhomentropic and nonisoenergetic flow which is specified by tabular input. A one-dimensional spline fit is used to produce the necessary continuous data specification. Here again a link for a user-supplied subprogram for initial data generation has been provided.

The output comprises velocity components and pressure at each of the computed intersections of the selected system of streamlines with each solution surface. A six-component thrust integration is performed at each solution surface. The amount of printed output can be varied to suit the needs of the user. The results which are presented in this paper have been summarized by using a Cal-Comp plotter to produce isometric streamline plots, nozzle cross-section plots, and polar wall pressure plots.

Numerical Results

The computer program has been used to generate solutions for source and simple-wave flows,[1,2] and the results have been compared to the exact solutions for the purpose of numerically establishing the order of accuracy as well as the absolute accuracy. These results demonstrated the second-order accuracy of the scheme, and in addition, demonstrated very good absolute accuracy, well within 1% for integration using practical step sizes. In addition, numerical results obtained with the program for two-dimensional axisymmetric flows were compared to solutions obtained using a conventional two-dimensional method of characteristics program.

Results for a highly contoured nozzle are discussed in Ref. 2; similar data for a 15° conical nozzle are compared in Fig. 1. The agreement in pressure is within 1% except in the vicinity of points of discontinuous rates of change, such as point A on the axis of symmetry where the first expansion wave from the circular arc throat contour reaches the axis. At this point, the two-dimensional solution shows the true discontinuous character in the slope of the axial pressure distribution, while the three-dimensional solution exhibits some smoothing of the sudden change. The smoothing is due to the interpolation process in which continuity of the dependent variables and their first derivatives is assumed. The diffusion of sudden changes is inherent in any numerical scheme for hyperbolic equations where the difference scheme zone of

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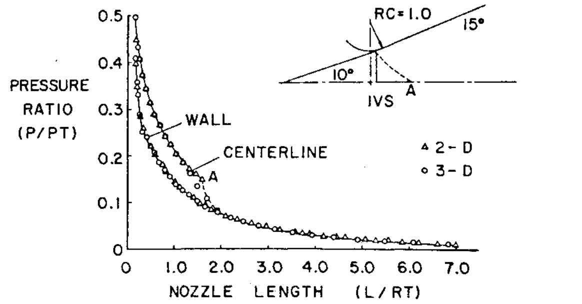

Fig. 1 Three-dimensional calculations for a 15° cone.

dependence is larger than the differential zone of dependence. This situation arises in the present case due to the necessity of satisfying the Courant-Freidrichs-Lewy stability condition, which requires that the speed of propagation of a disturbance through the difference network must everywhere equal or exceed the true sound speed.[19]

Elliptical Nozzles

Results for two elliptical nozzles are presented in Figs. 2 and 3. These nozzles have elliptical cross sections normal to the X-coordinate axis. The Y and Z intercepts of the cross sections are functions of the X-coordinate such that the contour is initially circular at the throat and elliptical beyond. The intercept variation is described by a circular arc in the throat region which is joined tangentially to a general parabola for the diverging section.

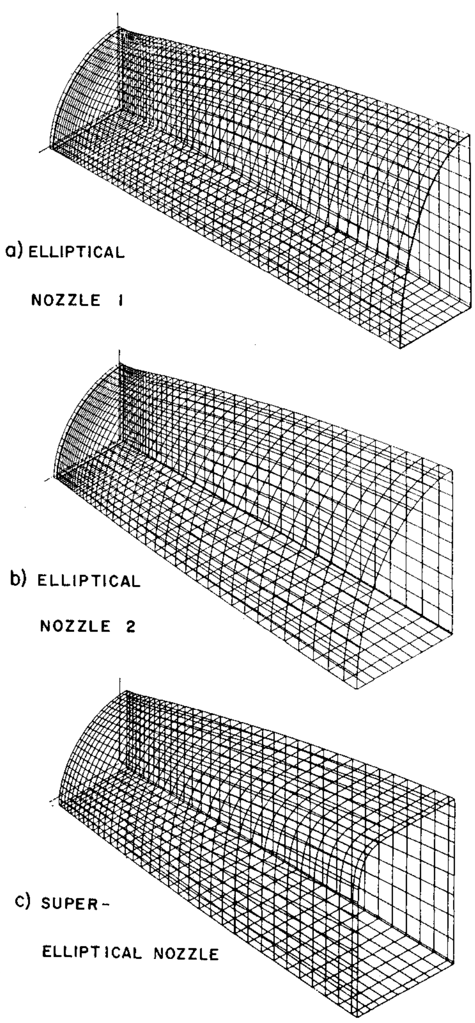

Fig. 2 Isometric plots of three nozzle contours.

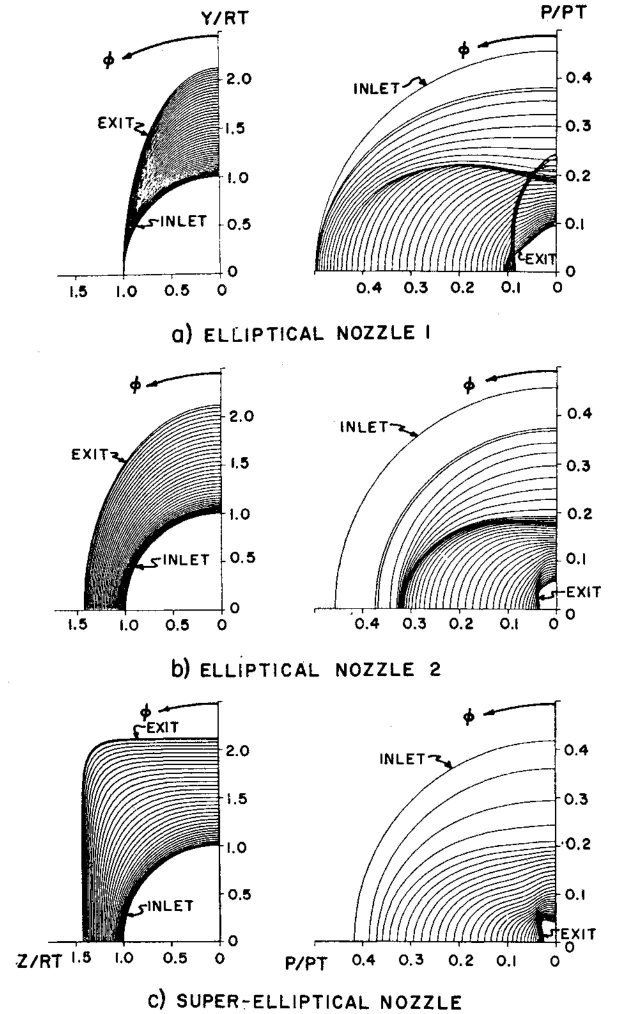

Fig. 3 Cross sections and polar pressure plots for nozzles of Fig. 2.

The cross sections and boundary streamlines for one quadrant of the first nozzle are plotted isometrically in Fig. 2a. In this case the intercept of the contour with the X-Z coordinate plane was held fixed while the X-Y intercept was allowed to vary. An axisymmetric source flow was used to establish flow conditions on the initial-value surface.

The nozzle cross sections and corresponding polar pressure plots at each solution surface are shown in Fig. 3a. Note the pronounced three-dimensional character of the pressure field. Even though the contour is a smooth and relatively gentle transition, significant transverse gradients in pressure are present. Clearly a pseudo three-dimensional calculation technique could not adequately predict such a flow.

The second elliptical nozzle also is circular at the throat and has variations of both the X-Y and X-Z intercepts in the diverging section. An isometric plot of the nozzle cross sections and the boundary streamlines is shown in Fig. 2b. The corresponding cross sections and polar pressure plots are shown in Fig. 3b. This nozzle has less deviation from

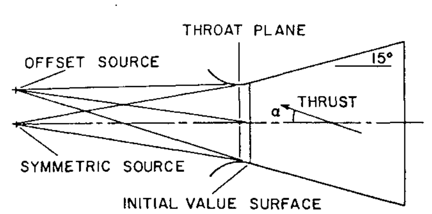

Fig. 4 Double source model.

axial symmetry than the first nozzle, and consequently, the three-dimensional character is less pronounced.

"Super-Elliptical" Nozzle

A "super-elliptical" nozzle contour was generated using the same intercepts as for the second elliptical nozzle, but letting the "super-elliptical" exponents vary from 2.0 at the throat station to 10.0 at the nozzle exit so that the contour is initially circular at the throat and becomes "super-elliptical" in the diverging section. One quadrant of the resulting contour is shown isometrically in Fig. 2c. The corresponding cross sections and polar pressure plots are shown in Fig. 3c. Here again, pronounced three-dimensional flow effects are present. Note that the highest pressures in the exit plane are in the corner while the lowest pressures occur at the minor axis intercept. A prediction based on a longitudinal two-dimensional expansion area ratio would produce just the opposite result. Again the rather complex three-dimensional character of the flow is evident.

Each of the three solutions presented in Figs. 2 and 3 were obtained using approximately 5000 mesh points within the flow. The computations were performed on the CDC 6500 computer and 15 min of central processor time were required in each case.

Nonsymmetric Inlet Flow

The effect of nonsymmetric inlet flow into an axisymmetric nozzle was investigated in order to illustrate the application of the method to a current engineering problem. Nonsymmetric inlet flow is frequently encountered in solid rocket motors having asymmetric grain configurations of multiple nozzle aft closures. The asymmetric inlet flow results in misalignment of the thrust axis and is a source of dispersion in unguided rockets.[20]

A nonsymmetric inlet flow was generated, for the purpose of these calculations, by superimposing two one-dimensional source flows, one located symmetrically and the other one offset from the axis. The two source flows were superimposed on the initial value surface by using periodic weighting functions. The symmetric source was weighted by the factor $\sin^2[(\pi/2)(R/RT)]$ and the asymmetric source was weighted by $\cos^2[(\pi/2)\ (R/RT)]$. The geometry of the double source model is illustrated in Fig. 4. The periodic weighting functions produce the desirable result that the flow is everywhere tangent to the nozzle boundary and has the character of the offset source at the center of the flow.

Figure 5 shows the wall pressure ratios at the upper and lower wall intersections with the plane of symmetry, plotted as a function of nozzle length. Note the reversal in relative magnitudes which occurs approximately midway down the nozzle. The Mach number at the center of the initial-value surface was 1.05 and the half-angle of the symmetric source was 5°. The offset angle of the second source was also 5°. The area ratio of the nozzle was 13.3.

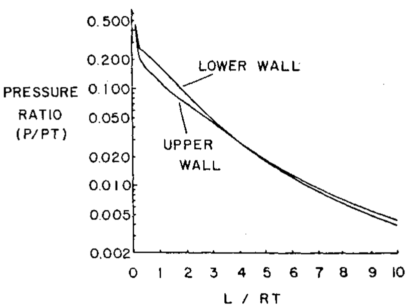

Fig. 5 Pressure plots for skewed inlet flow.

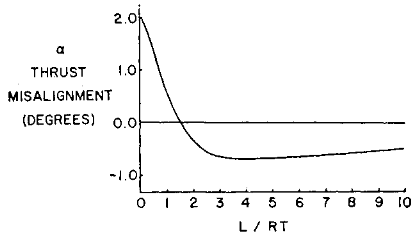

Fig. 6 Thrust misalignment for skewed inlet flow.

The angle of thrust misalignment α is presented in Fig. 6 as a function of nozzle length. The results show the damped periodic variation discussed in Ref. 20. Although the results show only one thrust reversal, the period of such reversals increases greatly with the degree of expansion and a considerably longer nozzle would be required to obtain a second reversal. The frequency of such reversals is a function of the cone angle or the rate of expansion.[20]

Approximately 8000 mesh points were used to obtain the skewed flow results, and the computations required 22 min of central processor time on the CDC 6500 computer.

Conclusions

It is now possible to examine analytically many three-dimensional internal flows which heretofore have been unsolvable. The method which was used to generate the solutions which are presented herein has been shown[1] to be accurate to second order, to be numerically stable, and to require reasonable computing times. The technique offers these same advantages for solution of many external flow problems, as well as other internal flow problems, such as supersonic inlets. The sample computations presented herein illustrate the complex nature of even modestly three-dimensional supersonic flows, and thus the inadequacy of pseudo three-dimensional calculations in which cross-flows are neglected.

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Boundary-Layer Analysis of Low-Density Nozzles, Including Displacement, Slip, and Transverse Curvature

David L. Whitfield*
ARO Inc., Arnold Air Force Station, Tenn.

AND

Clark H. Lewis†
Virginia Polytechnic Institute, Blacksburg, Va.

Boundary layers in low-density supersonic and hypersonic conical and contoured axisymmetric nozzles were investigated theoretically and comparisons have been made with available experimental data. The nozzle flow conditions treated were such that a thick laminar boundary layer existed on the wall and a small inviscid isentropic core existed along the center line. The nonsimilar laminar boundary-layer equations including first-order transverse curvature terms for internal (nozzle) flow were solved numerically with and without wall slip and temperature jump boundary conditions and boundary-layer displacement effects. Solutions are presented for five low-density nozzles whose exit conditions ranged in Mach number from about 3.0 to 10.0 and in *Re*/ft from about 350 to 15,000. Displacement and transverse curvature effects were significant; however, slip effects were found to be negligible. Agreement between numerical results and experimental data is good, particularly in the nozzle test sections.

Nomenclature

c_p = constant-pressure specific heat
g = total enthalpy ratio, H/H_o
H_o = total enthalpy in reservoir
h = $-q_w/(T_{aw} - T_w)$, heat-transfer coefficient
k = thermal conductivity
L = reference body length
M = Mach number
P, P_o, P'_o = static, total, and Pitot pressures, respectively
Pr = $\mu c_p/k$, Prandtl number
q_w = nozzle wall heat-transfer rate per unit area
R = converging section radius of curvature
Re = $\rho u/\mu$, Reynolds number per unit length
r = radius defined by Eq. (1)
r_w, r^* = nozzle wall and throat radii, respectively
St = $-q_w/\rho_e u_e H_o(1-g_w)$, Stanton number
T, T_0 = static and total temperatures, respectively
$T(0)$ = slip temperature at the wall
T_0' = total temperature behind a normal shock
T_{aw} = adiabatic wall temperature
u = velocity component in x direction
$u(0)$ = slip velocity at the wall
x = surface distance along nozzle wall
y = distance normal to nozzle wall
z = distance along nozzle axis
z_m = total distance along nozzle axis
α = nozzle wall angle
β = $(2\xi/u_e)du_e/d\xi$, dimensionless velocity gradient
δ = boundary-layer thickness (y where $u/u_e = 0.995$)
δ^* = displacement thickness
γ = ratio of specific heats
λ = mean free path

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* Engineer, Plume Test and Analysis Section, Testing and Development Branch, Aerospace Environmental Facility.

† Associate Professor of Aerospace Engineering; previously Supervisor, Theoretical Gas Dynamics Section, Hypervelocity Branch, Aerophysics Division, von Kármán Gas Dynamics Facility. Associate Fellow AIAA.